# Supplementary Material: Efficient Algorithms for Robust One-bit Compressive Sensing

**Lijun Zhang** ZHANGLJ LAMDA.NJU.EDU.CN
National Key Laboratory for Novel Software Technology, Nanjing University, Nanjing, China

**Jinfeng Yi** JINFENGY US.IBM.COM
IBM Thomas J. Watson Research Center, Yorktown Heights, NY 98, USA

**Rong Jin** RONGJIN CSE.MSU.EDU
Department of Computer Science and Engineering, Michigan State University, East Lansing, MI 48824, USA

## A. Proof of Lemma

We consider the following general optimization problem

$$\min_{\|\mathbf{x}\|_2\leq 1} -\mathbf{x}^\top \mathbf{y} + \gamma\|\mathbf{x}\|_1.$$

Before we proceed, we need the following lemma.

**Lemma 6.** *The solution to the optimization problem*

$$\min_x \frac{1}{2}(x-y)^2 + \gamma|x|$$

*is given by*

$$P_\gamma(y) = \begin{cases} 0, & \text{if } |y|\leq\gamma; \\ sign(y)(|y|-\gamma), & \text{otherwise.}\end{cases}$$

*where $P_\gamma(\cdot)$ is the soft-thresholding operator defined in (7) (Donoho, 1995).*

The proof of Lemma 6 can be found in Duchi Singer 9. Based on the above lemma, it is easy to verify that

$$\min_x \frac{1}{2}(x-y)^2 + \gamma|x| = \begin{cases} \frac{y^2}{2}, & \text{if } |y|\leq\gamma; \\ \gamma|y| - \frac{\gamma^2}{2}, & \text{otherwise.}\end{cases} \quad 6$$

First, we consider the case $\|y\|_\infty \leq \gamma$. Then, it is easy to verify that

$$\mathbf{0} \in \operatorname*{argmin}_{\mathbf{x}} -\mathbf{x}^\top\mathbf{y} + \gamma\|\mathbf{x}\|_1.$$

Since $\|\mathbf{0}\|_2 \leq 1$, $\mathbf{0}$ is also an optimal solution to .

Next, we consider the case $\|y\|_\infty > \gamma$. Following the standard analysis of convex optimization Boyd Vandenberghe 4, the Lagrange dual function $g(\mu)$ of is given by

$$\begin{aligned} & g(\mu) \\ =& \min_{\mathbf{x}} -\mathbf{x}^\top\mathbf{y} + \gamma\|\mathbf{x}\|_1 + \mu(\|\mathbf{x}\|_2^2 - 1) \\ =& \min_{\mathbf{x}} 2\mu\left(\frac{1}{2}\left\|\mathbf{x} - \frac{\mathbf{y}}{2\mu}\right\|_2^2 + \frac{\gamma}{2\mu}\|\mathbf{x}\|_1\right) - \frac{\|\mathbf{y}\|_2^2}{4\mu} - \mu \\ =& 2\mu\left(\sum_i \min_{x_i}\frac{1}{2}\left(x_i - \frac{y_i}{2\mu}\right)^2 + \frac{\gamma}{2\mu}|x_i|\right) - \frac{\|\mathbf{y}\|_2^2}{4\mu} - \mu \\ \overset{6}{=}& 2\mu\left(\sum_{i:|y_i|\leq\gamma}\frac{y_i^2}{8\mu^2} + \sum_{i:|y_i|>\gamma}\left(\frac{\gamma|y_i|}{4\mu^2} - \frac{\gamma^2}{8\mu^2}\right)\right) \\ & - \frac{\|\mathbf{y}\|_2^2}{4\mu} - \mu \\ =& \sum_{i:|y_i|>\gamma}\left(\frac{\gamma|y_i|}{2\mu} - \frac{\gamma^2}{4\mu} - \frac{y_i^2}{4\mu}\right) - \mu \\ =& -\frac{\sum_{i:|y_i|>\gamma}(|y_i|-\gamma)^2}{4\mu} - \mu = -\frac{\|P_\gamma(\mathbf{y})\|_2^2}{4\mu} - \mu. \end{aligned}$$

So, the Lagrange dual problem is

$$\max_{\mu\geq 0} -\frac{\|P_\gamma(\mathbf{y})\|_2^2}{4\mu} - \mu$$

and the optimal dual solution is

$$\mu_* = \frac{\|P_\gamma(\mathbf{y})\|_2}{2}.$$

Following the standard analysis Boyd Vandenberghe 4 Section , the optimal primal solution is

$$\begin{aligned} \mathbf{x}_* =& \operatorname*{argmin}_{\mathbf{x}} \frac{1}{2}\left\|\mathbf{x} - \frac{\mathbf{y}}{2\mu_*}\right\|_2^2 + \frac{\gamma}{2\mu_*}\|\mathbf{x}\|_1 \\ \overset{\text{Lemma 6}}{=}& \frac{1}{\|P_\gamma(\mathbf{y})\|_2}P_\gamma(\mathbf{y}). \end{aligned}$$

## B. Proof of Lemma

We first consider the case $\mathrm{sign}(\mathbf{x}_k^\top \mathbf{u}) = 1$, i.e.,

$$\mathbf{x}_k^\top \frac{\mathbf{u}}{\|\mathbf{u}\|_2} > \delta_k.$$

Then, we have

$$\begin{aligned}\mathbf{x}_*^\top \frac{\mathbf{u}}{\|\mathbf{u}\|_2} &= \mathbf{x}_k^\top \frac{\mathbf{u}}{\|\mathbf{u}\|_2} + (\mathbf{x}_* - \mathbf{x}_k)^\top \frac{\mathbf{u}}{\|\mathbf{u}\|_2} \\ &> \delta_k - \|\mathbf{x}_* - \mathbf{x}_k\|_2 \geq 0.\end{aligned}$$

Thus

$$\mathrm{sign}(\mathbf{x}_*^\top \mathbf{u}) = \mathrm{sign}\left(\mathbf{x}_*^\top \frac{\mathbf{u}}{\|\mathbf{u}\|_2}\right) = 1 = \mathrm{sign}(\mathbf{x}_k^\top \mathbf{u})$$