Improved Dynamic Regret for Online Frank-Wolfe

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Abstract

To deal with non-stationary online problems with complex constraints, we investigate the dynamic regret of online Frank-Wolfe (OFW), which is an efficient projection-free algorithm for online convex optimization. It is well-known that in the setting of offline optimization, the smoothness of functions and the strong convexity of functions accompanying specific properties of constraint sets can be utilized to achieve fast convergence rates for the Frank-Wolfe (FW) algorithm. However, for OFW, previous studies only establish a dynamic regret bound of $O(\sqrt{T}(V_T + \sqrt{D_T} + 1))$ by utilizing the convexity of problems, where T is the number of rounds, V_T is the function variation, and D_T is the gradient variation. In this paper, we derive improved dynamic regret bounds for OFW by extending the fast convergence rates of FW from offline optimization to online optimization. The key technique for this extension is to set the step size of OFW with a line search rule. In this way, we first show that the dynamic regret bound of OFW can be improved to $O(\sqrt{T(V_T + 1)})$ for smooth functions. Second, we achieve a better dynamic regret bound of $O(T^{1/3}(V_T + 1)^{2/3})$ when functions are smooth and strongly convex, and the constraint set is strongly convex. Finally, for smooth and strongly convex functions with minimizers in the interior of the constraint set, we demonstrate that the dynamic regret of OFW reduces to $O(V_T + 1)$, and can be further strengthened to $O(\min\{P_T^*, S_T^*, V_T\} + 1)$ by performing a constant number of FW iterations per round, where P_T^* and S_T^* denote the path length and squared path length of minimizers, respectively.

Keywords: Online Convex Optimization, Dynamic Regret, Frank-Wolfe Algorithm

1. Introduction

Online convex optimization (OCO) is a powerful learning paradigm, which can be utilized to model a wide variety of machine learning problems, and is commonly formulated as a repeated game over a convex constraint set $K \subseteq \mathbb{R}^d$ (Hazan, 2016). In each round $t = 1, \ldots, T$, a player first selects a feasible decision $\mathbf{x}_t \in K$, and then suffers a loss $f_t(\mathbf{x}_t)$, where $f_t(\mathbf{x}) : K \mapsto \mathbb{R}$ is a convex function and could be selected in the adversarial way. The goal of the player is to minimize the cumulative loss $\sum_{t=1}^{T} f_t(\mathbf{x}_t)$. A standard algorithm is online gradient descent (OGD) (Zinkevich, 2003), which usually has both theoretical and practical appeals. However, to ensure the feasibility of decisions, OGD needs to perform a projection operation per round, which could be computationally expensive in high-dimensional problems with complex constraints and thus limits its applications (Hazan and Kale, 2012). To tackle this issue, Hazan and Kale (2012) propose the first projection-free algorithm for OCO, namely online Frank-Wolfe (OFW), by eschewing the projection operation with one iteration of the Frank-Wolfe (FW) algorithm (Frank and Wolfe, 1956; Jaggi, 2013), which is much more efficient for many complex constraint sets.

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Table 1: Summary of Convergence Rates of FW with T iterations in Offline Optimization. Abbreviations: convex ! cvx, strongly convex ! scvx, smooth ! sm.

Reference	Function	K	Location of Minimizers	Convergence Rates
Jaggi (2013)	sm & cvx	cvx	unrestricted	$O(1=T)$
Garber and Hazan (2015)	sm & scvx	scvx	unrestricted	$O(1=T^2)$
Garber and Hazan (2015)	sm & scvx	cvx	the interior of K	$O(\exp(-T))$

Attracted by this computational advantage, there has been a growing research interest in developing and analyzing efficient projection-free online algorithms under different scenarios (Garber and Hazan, 2016; Levy and Krause, 2019; Hazan and Minasyan, 2020; Wan and Zhang, 2021; Wan et al., 2020, 2022a,b; Kalhan et al., 2021; Garber and Kretzu, 2021, 2022; Mhammedi, 2022a,b; Lu et al., 2023; Zhou et al., 2023). However, most of them focus on minimizing the regret

$$\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \min_{\mathbf{x} 2 K} \sum_{t=1}^{T} f_t(\mathbf{x}) \tag{1}$$

which compares the player against a fixed comparator, and thus cannot reflect the hardness of problems with non-stationary environments, where the best decision could be time-varying. To address this limitation, we investigate efficient projection-free algorithms with a more suitable metric called dynamic regret (Zinkevich, 2003)

$$\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}_t) \tag{2}$$

which compares the player against the minimizer $\mathbf{x}_t$ 2 $\operatorname{argmin}_{\mathbf{x} 2 K} f_t(\mathbf{x})$ in each round t. Although a few studies (Kalhan et al., 2021; Zhou et al., 2023) have proposed to minimize the dynamic regret with OFW (Hazan and Kale, 2012), current theoretical understandings of this way are limited.

To be precise, Kalhan et al. (2021) for the first time prove that OFW can attain a dynamic regret bound of $O(\sqrt{T}(V_T + \sqrt{D_T} + 1))$ for smooth functions by directly applying the FW iteration to the function $f_t(\mathbf{x})$, where $V_T = \sum_{t=2}^{T} \max_{\mathbf{x} 2 K} j f_t(\mathbf{x}) - f_{t-1}(\mathbf{x}) j$ and $D_T = \sum_{t=2}^{T} k r f_t(\mathbf{x}_t) - r f_{t-1}(\mathbf{x}_{t-1}) k_2^2$ denote the function variation and the gradient variation, respectively. However, very recently, Zhou et al. (2023) show that OFW actually can attain the $O(\sqrt{T}(V_T + \sqrt{D_T} + 1))$ dynamic regret bound by only utilizing the convexity of problems, which implies that the smoothness of functions is not appropriateTd [(t)]TJ/F21 [(x)]TJ/F24 1 239701 Tf 6.364 -1.68hbe oNoti2.363 cm(attain.716 Tf 16

which improves the existing $O(\sqrt{T}(V_T + \sqrt{D_T} + 1))$ bound (Kalhan et al., 2021; Zhou et al., 2023) by removing the dependence on D_T and reducing the dependence on V_T. The insight behind these improvements is that by refining the existing analysis of OFW for smooth functions (Kalhan et al., 2021), the dynamic regret of OFW with a fixed step size σ actually is $O(\sigma^{-1}(V_T + 1) + \sigma T)$, which does not depend on D_T and motivates us to minimize the dependence on V_T by adjusting the step size. The key technical challenge is that the value of V_T actually is unknowable in practice, which limits the choice of the step size. Inspired by previous studies on OCO (van Erven and Koolen, 2016; Zhang et al., 2018a), a standard way to address this limitation is to run multiple OFW, each with a different step size depending on an estimated V_T, and combine them via an expert-tracking algorithm. However, it requires $O(\log T)$ instances of OFW, which clearly increases the computational complexity and is unacceptable for real applications with a large T. By contrast, we adopt a simple line search rule (Garber and Hazan, 2015) to select the step size of OFW, which can be implemented as efficient as the original OFW.

Furthermore, if functions are smooth and strongly convex, we prove that the line search rule enables OFW to automatically reduce the dynamic regret to $O(T^{1/3}(V_T + 1)^{2/3})$ over strongly convex sets, and $O(V_T + 1)$ over convex sets in case the minimizers of functions lie in the interior of the set. These two improvements are analogous to the improvements in the convergence rate of FW for offline optimization. Finally, we demonstrate that under the same assumptions, the $O(V_T + 1)$ dynamic regret can be further strengthened to $O(\min\{P_T, S_T, V_T\} + 1)$ by performing a constant number of FW iterations per round, where $P_T = \sum_{t=2}^{T} \|\mathbf{x}_t - \mathbf{x}_{t-1}\|_2$]TJ/F74 11.9091 Tf 7.815 pnd

been improved to some data-dependent terms for convex functions (Cutkosky, 2020) and smooth functions (Zhao et al., 2020). Furthermore, Baby and Wang (2021) consider exponentially concave functions and strongly convex functions, and respectively establish a general dynamic regret bound of $\mathcal{O}(d^{3.5}(T^{1/3}C_T^{2/3} + 1))$ and an improved one of $\mathcal{O}(d^2(T^{1/3}C_T^{2/3} + 1))$,[1] where d is the dimensionality of decisions and $C_T = \sum_{t=2}^{T} \|\mathbf{u}_t - \mathbf{u}_{t-1}\|_1$.

Notice that the dynamic regret defined in (2) can be regarded as the worst case of the general dynamic regret defined in (3), which allows us to derive worst-case bounds by combining the above results with $\mathbf{u}_t = \mathbf{x}_t$. However, this way may not lead to the best results for the dynamic regret defined in (2), since it ignores the specific property of $\mathbf{x}_t$. For this reason, there exist plenty of studies that focus on the dynamic regret defined in (2). If each function $f_t(\mathbf{x})$ is smooth and its minimizer $\mathbf{x}_t$ lies in the interior of $\mathcal{K}$, Yang et al. (2016) show that OGD can achieve an $O(P_T + 1)$ dynamic regret bound. When functions are smooth and strongly convex, Mokhtari et al. (2016) establish the same dynamic regret bound for OGD. Moreover, when functions are smooth and strongly convex, and their minimizers lie in the interior of $\mathcal{K}$, Zhang et al. (2017) propose the online multiple gradient descent (OMGD) algorithm, and improve the dynamic regret to $O(\min\{P_T, S_T\} + 1)$. Besides, Besbes et al. (2015) prove that a restarted variant of OGD can attain $O(T^{2/3}(V_T + 1)^{1/3})$ and $\mathcal{O}(\sqrt{T(V_T + 1)})$ dynamic regret bounds for convex and strongly convex functions respectively, while they need to know the value of V_T beforehand. By applying a strongly adaptive online learning framework (Hazan and Seshadhri, 2007; Daniely et al., 2015), in the case without the value of V_T, Zhang et al. (2018b) establish $\mathcal{O}(T^{2/3}(V_T + 1)^{1/3})$, $\mathcal{O}(d\sqrt{T(V_T + 1)})$, and $\mathcal{O}(\sqrt{T(V_T + 1)})$ dynamic regret bounds for convex functions, exponentially concave functions, and strongly convex functions, respectively. Under the same assumption as Zhang et al. (2017), Zhao and Zhang (2021) recently refine the dynamic regret of OMGD to $O(\min\{P_T, S_T, V_T\} + 1)$.[2] They also demonstrate that a greedy strategy that sets $\mathbf{x}_{t+1} = \mathbf{x}_t$ can achieve the same dynamic regret bound without the strong convexity of functions.

2.2. Dynamic Regret of Projection-free Algorithms

However, the above algorithms require either the projection operation or more complicated computations, which cannot efficiently deal with complex constraints. To tackle this issue, Wan et al. (2021) propose an online algorithm that only utilizes FW iterations to update the decision, and establish $O(T^{2/3}(V_T + 1)^{1/3})$ and $\mathcal{O}(\sqrt{T(V_T + 1)})$ dynamic regret bounds for convex and strongly convex functions, respectively. Unfortunately, the algorithm needs to utilize $\mathcal{O}(T)$ and $\mathcal{O}(T^2)$ FW iterations per round for achieving these two bounds, which actually suffers a similar computation complexity compared with the projection operation. By contrast, Kalhan et al. (2021) prove that by directly applying the FW iteration to the function $f_t(\mathbf{x})$, OFW (Hazan and Kale, 2012) can attain an $O(\sqrt{T(V_T + 1)} + \sqrt{D_T + 1})$ dynamic regret bound for smooth functions. Recently, Zhou et al. (2023) show that the same dynamic regret bound can be achieved by OFW without the smoothness of functions.

2.3. Discussions

As previously mentioned, if considering the regret defined in (1) rather than the dynamic regret, there exist plenty of projection-free online algorithms for different scenarios (Garber and Hazan,

1. The $\mathcal{O}$ notation hides constant factors as well as polylogarithmic factors in T.
2. The special location of the minimizer is only required to achieve the upper bound in terms of S_T^*.

Table 2: Comparison of our results to previous studies. Here, we only summarize algorithms, which utilize at most $O(1)$ projections or FW iterations per round, and achieve dynamic regret bounds that are comparable with our results, i.e., depending on V_T. Abbreviations: convex $\to$ cvx, strongly convex $\to$ scvx, smooth $\to$ sm, interior$(\mathcal{K}) \to$ the interior of $\mathcal{K}$, # PROJ $\to$ numbers of projections per round, # FW $\to$ numbers of FW iterations per round.

Assumptions	Reference	# PROJ	# FW	Dynamic Regret
$f_t(\cdot)$: cvx $\mathcal{K}$: cvx	Zhang et al. (2018b)	$O(1)$	0	$O(T^{2/3}(V_T+1)^{1/3})$
	Zhou et al. (2023)	0	1	$O(\sqrt{T}(V_T+\sqrt{D_T}+1))$
$f_t(\cdot)$: sm & cvx $\mathcal{K}$: cvx	Kalhan et al. (2021)	0	1	$O(\sqrt{T}(V_T+\sqrt{D_T}+1))$
	Theorem 2	0	1	$O(\sqrt{T(V_T+1)})$
$f_t(\cdot)$: sm & scvx $\mathcal{K}$: scvx	Zhao and Zhang (2021)	$O(1)$	0	$O(\min\{P_T, V_T\}+1)$
	Theorem 3	0	1	$O(T^{1/3}(V_T+1)^{2/3})$
$f_t(\cdot)$: sm & scvx $\mathcal{K}$: cvx $x_t \in \text{interior}(\mathcal{K})$	Zhao and Zhang (2021)	$O(1)$	0	$O(\min\{P_T, S_T, V_T\}+1)$
	Theorem 4	0	1	$O(V_T+1)$
	Theorem 5	0	$O(1)$	$O(\min\{P_T, S_T, V_T\}+1)$

2016; Levy and Krause, 2019; Hazan and Minasyan, 2020; Wan and Zhang, 2021; Wan et al., 2020, 2022a,b; Garber and Kretzu, 2021, 2022; Mhammedi, 2022a,b; Lu et al., 2023). Although both the dynamic regret and projection-free algorithms have attracted much attention, OFW is the only efficient projection-free algorithm for minimizing the dynamic regret. In this paper, we improve the dynamic regret of OFW by extending the fast convergence rates of FW from offline optimization to online optimization. Moreover, it is worth noting that as summarized in Table 2, even compared with projection-based algorithms, our results can match the best $O(\min\{P_T, S_T, V_T\}+1)$ dynamic regret bound under the same assumptions, and exploit the smoothness of functions to improve the $O(T^{2/3}(V_T+1)^{1/3})$ dynamic regret bound achieved by only utilizing the convexity of functions.[3]

3. Main Results

We first revisit OFW for smooth functions, and then show the advantage of utilizing a line search rule. Finally, for smooth and strongly convex functions with minimizers in the interior of the set, we propose OFW with multiple updates to further strengthen the performance.

3.1. Revisiting Online Frank-Wolfe for Smooth Functions

To minimize the dynamic regret, Kalhan et al. (2021) propose a specific instance of OFW, which first selects an arbitrary $x_1 \in \mathcal{K}$ and then iteratively updates the decision as follows

$$v_t \in \operatorname*{argmin}_{x \in \mathcal{K}} \langle \nabla f_t(x_t), x \rangle, \quad x_{t+1} = (1-\rho)x_t + \rho v_t \tag{4}$$

3. In Appendix H, we show that OGD (Zinkevich, 2003) can also achieve $O(\sqrt{T(V_T+1)})$ dynamic regret bound for smooth functions, which may be of independent interest.

Theorem 2 *Under Assumptions 1 and 2, Algorithm 1 ensures*

$$\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}_t) \leq \sqrt{MT(V_T + M)} + \frac{\alpha D^2}{2}\sqrt{\frac{(V_T + M)T}{M}}$$

where $V_T = \sum_{t=2}^{T} \max_{\mathbf{x} 2 K} j f_t(\mathbf{x}) - f_{t-1}(\mathbf{x})j$ *and* $M = \max_{t 2 [T]; \mathbf{x} 2 K} f2j f_t(\mathbf{x})jg.$

Remark. Theorem 2 implies that by utilizing the line search rule, OFW can attain a dynamic regret bound of $O(\sqrt{T(V_T + 1)})$ for smooth functions without the prior information of V_T. Compared with the $O(\sqrt{T(V_T + \sqrt{D_T} + 1)})$ bound achieved by Kalhan et al. (2021), this bound removes the dependence on D_T and reduces the dependence on V_T.

Additionally, during the analysis of Lemma 3 and Theorem 2, we realize that the convergence property of one FW iteration is critical for upper bounding $f_{t+1}(\mathbf{x}_{t+1}) - f_{t+1}(\mathbf{x}_{t+1})$, and then affects the dynamic regret. This motivates us to further investigate the dynamic regret of Algorithm 1 in the following two special cases, in which the FW algorithm has been proved to converge faster (Garber and Hazan, 2015).

3.2.1. STRONGLY CONVEX FUNCTIONS AND SETS

To be precise, in the first case, we assume that these smooth functions are also strongly convex, and the decision set is strongly convex.

Assumption 3 *All loss functions are* β_f*-strongly convex, i.e., for any* $t 2 [T]; \mathbf{x} 2 K; \mathbf{y} 2 K$*, it holds that*

$$f_t(\mathbf{y}) \geq f_t(\mathbf{x}) + h r f_t(\mathbf{x}); \mathbf{y} - \mathbf{x}i + \frac{\beta_f}{2}k\mathbf{x} - \mathbf{y}k_2^2:$$

Assumption 4 *The decision set is* β_K*-strongly convex, i.e., for any* $\mathbf{x} 2 K; \mathbf{y} 2 K; \gamma 2 [0; 1]$ *and* $\mathbf{z} 2 \mathbb{R}^d$ *such that* $k\mathbf{z}k_2 = 1$*, it holds that*

$$\gamma\mathbf{x} + (1 - \gamma)\mathbf{y} + \gamma(1 - \gamma)\frac{\beta_K}{2}k\mathbf{x} - \mathbf{y}k_2^2\mathbf{z} 2 K:$$

By further incorporating these two assumptions, we establish the following lemma, which significantly improves Lemma 3 by removing the $\frac{\alpha\sigma_s^2 D^2}{2}$ term in the right side of (8).

Lemma 4 *Under Assumptions 1, 2, 4, for* $t = 1; \cdots; T - 1$*, Algorithm 1 ensures*

$$f_{t+1}(\mathbf{x}_{t+1}) - f_{t+1}(\mathbf{x}_{t+1}) \leq \max_{\mathbf{x} 2 K} j f_{t+1}(\mathbf{x}) - f_t(\mathbf{x})j + C_t(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)) + f_t(\mathbf{x}_t) - f_{t+1}(\mathbf{x}_{t+1})$$

where $C_t = \max\left\{1 - \frac{\beta_K k r f_t(\mathbf{x}_t)k_2}{8\alpha}; \frac{1}{2}\right\}.$

Based on Lemma 4, we establish the following lemma.

Theorem 3 *Under Assumptions 1, 2, 3, and 4, Algorithm 1 ensures*

$$\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}_t) \leq \left(\frac{8\sqrt{2\alpha}}{\sqrt{\beta_f \beta_K}}(M + V_T)\right)^{2=3} T^{1=3} + 2(M + V_T)$$

where $V_T = \sum_{t=2}^{T} \max_{\mathbf{x} 2 K} j f_t(\mathbf{x}) - f_{t-1}(\mathbf{x})j$ *and* $M = \max_{t 2 [T]; \mathbf{x} 2 K} f2j f_t(\mathbf{x})jg.$

Remark. From Theorem 3, Algorithm 1 can further exploit the strong convexity of functions and the set to achieve a dynamic regret bound of $O(T^{1=3}$

3.2.2. STRONGLY CONVEX FUNCTIONS WITH SPECIAL MINIMIZERS

Then, we proceed to consider the second case, in which all functions are strongly convex and their minimizers lie in the interior of the decision set. To this end, we further introduce the following assumption.

Assumption 5 *There exists a minimizer $\mathbf{x}_t \in \operatorname{argmin}_{\mathbf{x}\in K} f_t(\mathbf{x})$ that lies in the interior of K for any $t \in [T]$, i.e., there exists a parameter $r > 0$ such that if any $\mathbf{x}$ satisfies $\|\mathbf{x} - \mathbf{x}_t\|_2 \leq r$, then $\mathbf{x} \in K$.*

By replacing Assumption 4 in Lemma 4 with Assumptions 3 and 5, we establish the following lemma.

Lemma 5 *Under Assumptions 1, 2, 3, and 5, for $t = 1, \cdots, T-1$, Algorithm 1 ensures*

$$f_{t+1}(\mathbf{x}_{t+1}) - f_{t+1}(\mathbf{x}_{t+1}) \leq \max_{\mathbf{x}\in K} |f_{t+1}(\mathbf{x}) - f_t(\mathbf{x})| + C(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)) + f_t(\mathbf{x}_t) - f_{t+1}(\mathbf{x}_{t+1})$$

where $C = 1 - \frac{\beta_f r^2}{4\alpha D^2}$, $r = \min\left\{r, \frac{\sqrt{2\alpha D^2}}{\beta_f M}\right\}$, and $M = \max_{t\in[T],\mathbf{x}\in K} |f_t(\mathbf{x})|$.

Lemma 5 improves Lemma 4 by replacing the variable C_t with a constant C. Moreover, we derive the following theorem.

Theorem 4 *Under Assumptions 1, 2, 3, and 5, Algorithm 1 ensures*

$$\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}_t) \leq \frac{4\alpha(M + V_T)D^2}{\beta_f r^2}$$

where $M = \max_{t\in[T],\mathbf{x}\in K} |f_t(\mathbf{x})|$, $r = \min\left\{r, \frac{\sqrt{2\alpha D^2}}{\beta_f M}\right\}$, and $V_T = \sum_{t=2}^{T} \max_{\mathbf{x}\in K} |f_t(\mathbf{x}) - f_{t-1}(\mathbf{x})|$.

Remark. Theorem 4 implies that Algorithm 1 can achieve an $O(V_T + 1)$ dynamic regret bound over convex sets by further exploiting the strong convexity of functions and the special location of minimizers. This bound is better than the $O(T^{1/3}(V_T + 1)^{2/3})$ bound given by Theorem 3, which implies that the special location of minimizers is more useful for OFW than the strong convexity of sets.

3.3. Online Frank-Wolfe with Multiple Updates

Furthermore, we note that under the same assumptions as in our Theorem 4, previous studies (Zhao and Zhang, 2021) have achieved an $O(\min\{P_T, S_T, V_T\}+1)$ dynamic regret bound, which is tighter than the $O(V_T + 1)$ bound achieved by Algorithm 1. To fill this gap, we strengthen Algorithm 1 by performing a small number of FW iterations per round. Intuitively, more FW iterations will result in smaller $f_{t+1}(\mathbf{x}_{t+1}) - f_{t+1}(\mathbf{x}_{t+1})$, and thus reduce the dynamic regret. Specifically, the detailed procedures are outlined in Algorithm 2, which is named as online Frank-Wolfe with multiple updates and enjoys the following theoretical guarantee.

Algorithm 2 Online Frank-Wolfe with Multiple Updates

Input: the iteration number K per round
Initialization: $\mathbf{x}_1$ 2 K
for $t = 1; \cdots; T$ **do**
  $\mathbf{z}_{t+1}^0 = \mathbf{x}_t$
  for $i = 0; \cdots; K-1$ **do**
    Compute $\mathbf{v}_t^i$ 2 $\operatorname{argmin}_{\mathbf{x}2K} h\, r\, f_t(\mathbf{z}_{t+1}^i); \mathbf{x} i$
    Compute $\sigma_t^i = \operatorname{argmin}_{\sigma 2[0;1]} \sigma h\, r\, f_t(\mathbf{z}_{t+1}^i); \mathbf{v}_t^i - \mathbf{z}_{t+1}^i i + \frac{\alpha\sigma^2}{2} k\mathbf{z}_{t+1}^i - \mathbf{v}_t^i k_2^2$
    Update $\mathbf{z}_{t+1}^{i+1} = (1-\sigma_t^i)\mathbf{z}_{t+1}^i + \sigma_t^i \mathbf{v}_t^i$
  end for
  $\mathbf{x}_{t+1} = \mathbf{z}_{t+1}^K$
end for

Theorem 5 *Under Assumptions 1, 2, 3, and 5, by setting* $K = \left\lceil \frac{\ln(\beta_f r 4\alpha)}{\ln C} \right\rceil$ *where* $C = 1 - \frac{\beta_f r^2}{4\alpha D^2}$, $r = \min\left\{ r; \sqrt{\frac{2\alpha D^2}{\beta_f M}} \right\}$, *and* $M = \max_{t2[T]; \mathbf{x}2K} f2 j f_t(\mathbf{x}) j g$, *Algorithm 2 ensures*

$$\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}_t) \le \min\left\{ \frac{4\alpha(M + V_T)}{4\alpha - \beta_f}; 2GD + 2GP_T; \alpha D^2 + 2\alpha S_T \right\}$$

where $G = \max_{t2[T]; \mathbf{x}2K} k\, r\, f_t(\mathbf{x}_t) k_2$, $V_T = \sum_{t=2}^{T} \max_{\mathbf{x}2K} j f_t(\mathbf{x}) - f_{t-1}(\mathbf{x}) j$, $P_T = \sum_{t=2}^{T} k\mathbf{x}_t - \mathbf{x}_{t-1} k_2$, *and* $S_T = \sum_{t=2}^{T} k\mathbf{x}_t - \mathbf{x}_{t-1} k_2^2$.

Remark. From Theorem 5, for smooth and strongly convex functions with minimizers in the interior of the set, our Algorithm 2 can also achieve the $O(\min f P_T; S_T; V_T g + 1)$ dynamic regret bound by using a constant number of FW iterations per round, which strengthens the $O(V_T + 1)$ bound achieved by only using one FW iteration per round.

4. Theoretical Analysis

In this section, we only prove Lemma 3, Theorem 2, and Theorem 4. The omitted proofs are provided in the appendix. Compared with the existing analysis of OFW (Kalhan et al., 2021), the main novelty of our analysis is based on an appropriate way to incorporate the convergence property of one FW iteration (Garber and Hazan, 2015).

4.1. Proof of Lemma 3

We first decompose $f_{t+1}(\mathbf{x}_{t+1}) - f_{t+1}(\mathbf{x}_{t+1})$ as follows

$$\begin{aligned} f_{t+1}(\mathbf{x}_{t+1}) - f_{t+1}(\mathbf{x}_{t+1}) &= f_{t+1}(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_{t+1}) + f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t) + f_t(\mathbf{x}_t) - f_{t+1}(\mathbf{x}_{t+1}) \\ &\le \max_{\mathbf{x}2K} j f_{t+1}(\mathbf{x}) - f_t(\mathbf{x}) j + f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t) + f_t(\mathbf{x}_t) - f_{t+1}(\mathbf{x}_{t+1}): \end{aligned} \tag{9}$$

Then, we only need to analyze $f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t)$. Moreover, because $\mathbf{x}_{t+1}$ is generated by applying one FW iteration on $f_t(\mathbf{x}_t)$, it is natural to bound $f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t)$ by exploiting the convergence property of one FW iteration.

Specifically, by using the smoothness of $f_t(\mathbf{x})$, we have

$$f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t) \leq f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t) + \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_{t+1} - \mathbf{x}_t \rangle + \frac{\alpha}{2}\|\mathbf{x}_{t+1} - \mathbf{x}_t\|_2^2.$$

Moreover, according to Algorithm 1, we have

$$\mathbf{x}_{t+1} - \mathbf{x}_t = \sigma_t(\mathbf{v}_t - \mathbf{x}_t).$$

Combining the above two equations, we have

$$\begin{aligned} f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t) &\leq f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t) + \sigma_t \langle \nabla f_t(\mathbf{x}_t), \mathbf{v}_t - \mathbf{x}_t \rangle + \frac{\alpha\sigma_t^2}{2}\|\mathbf{v}_t - \mathbf{x}_t\|_2^2 \\ &\overset{(7)}{\leq} f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t) + \sigma \langle \nabla f_t(\mathbf{x}_t), \mathbf{v}_t - \mathbf{x}_t \rangle + \frac{\alpha\sigma^2}{2}\|\mathbf{v}_t - \mathbf{x}_t\|_2^2 \end{aligned} \tag{10}$$

for all possible $\sigma \in [0, 1]$.

Because of $\mathbf{v}_t \in \operatorname{argmin}_{\mathbf{x}\in\mathcal{K}} \langle \nabla f_t(\mathbf{x}_t), \mathbf{x} \rangle$ in Algorithm 1, we also have

$$\begin{aligned} \langle \nabla f_t(\mathbf{x}_t), \mathbf{v}_t - \mathbf{x}_t \rangle &\leq \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}_t \rangle \\ &\leq f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t) \end{aligned} \tag{11}$$

where the last inequality is due to the convexity of $f_t(\mathbf{x})$.

Then, combining (10) with (11), we have

$$\begin{aligned} f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t) &\leq (1 - \sigma)(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)) + \frac{\alpha\sigma^2}{2}\|\mathbf{v}_t - \mathbf{x}_t\|_2^2 \\ &\leq (1 - \sigma)(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)) + \frac{\alpha\sigma^2 D^2}{2} \end{aligned} \tag{12}$$

where the last inequality is due to Assumption 2.

Finally, we complete this proof by substituting (12) into (9).

4.2. Proof of Theorem 2

Let $R_T^D = \sum_{t=1}^T f_t(\mathbf{x}_t) - \sum_{t=1}^T f_t(\mathbf{x}_t)$. We first have

$$R_T^D = \sum_{t=2}^T f_t(\mathbf{x}_t) - \sum_{t=2}^T f_t(\mathbf{x}_t) + f_1(\mathbf{x}_1) - f_1(\mathbf{x}_1) \tag{13}$$

By using Lemma 3, for any $\sigma \in [0, 1]$, we have

$$\begin{aligned} R_T^D &\leq f_1(\mathbf{x}_1) - f_1(\mathbf{x}_1) + \sum_{t=2}^T \max_{\mathbf{x}\in\mathcal{K}} |f_t(\mathbf{x}) - f_{t-1}(\mathbf{x})| + \sum_{t=2}^T (1 - \sigma)(f_{t-1}(\mathbf{x}_{t-1}) - f_{t-1}(\mathbf{x}_{t-1})) \\ &\quad + \sum_{t=2}^T \left(f_{t-1}(\mathbf{x}_{t-1}) - f_t(\mathbf{x}_t) + \frac{\alpha\sigma^2 D^2}{2} \right) \\ &= f_1(\mathbf{x}_1) - f_1(\mathbf{x}_1) + V_T - (1 - \sigma)(f_T(\mathbf{x}_T) - f_T(\mathbf{x}_T)) + (1 - \sigma)R_T^D \\ &\quad + f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + \frac{\alpha(T-1)\sigma^2 D^2}{2} \\ &\leq f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T + \frac{\alpha(T-1)\sigma^2 D^2}{2} + (1 - \sigma)R_T^D \end{aligned}$$

where the last inequality is due to $f_T(\mathbf{x}_T) - f_T(\mathbf{x}_T) \geq 0$ and $\sigma \in [0,1]$.

Then, we have

$$\begin{aligned}\sigma R_T^D &\leq f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T + \frac{\alpha(T-1)\sigma^2 D^2}{2} \\ &\leq M + V_T + \frac{\alpha(T-1)\sigma^2 D^2}{2}\end{aligned}$$

where the last inequality is due to the definition of M.

By dividing both sides of the above inequality with σ and setting $\sigma = \sqrt{\frac{V_T+M}{MT}}$, we have

$$\begin{aligned}R_T^D &\leq \frac{M + V_T}{\sigma} + \frac{\alpha(T-1)\sigma D^2}{2} \\ &\leq \sqrt{MT(V_T+M)} + \frac{\alpha D^2}{2}\sqrt{\frac{(V_T+M)T}{M}}.\end{aligned}$$

4.3. Proof of Theorem 4

Let $R_T^D = \sum_{t=1}^T f_t(\mathbf{x}_t) - \sum_{t=1}^T f_t(\mathbf{x}_t)$. Combining (13) with Lemma 5, we have

$$\begin{aligned}R_T^D &\leq f_1(\mathbf{x}_1) - f_1(\mathbf{x}_1) + \sum_{t=2}^T \max_{\mathbf{x}\in K} \|f_t(\mathbf{x}) - f_{t-1}(\mathbf{x})\| + \sum_{t=2}^T C(f_{t-1}(\mathbf{x}_{t-1}) - f_{t-1}(\mathbf{x}_{t-1})) \\ &\quad + \sum_{t=2}^T \left(f_{t-1}(\mathbf{x}_{t-1}) - f_t(\mathbf{x}_t)\right) \\ &= f_1(\mathbf{x}_1) - f_1(\mathbf{x}_1) + V_T - C(f_T(\mathbf{x}_T) - f_T(\mathbf{x}_T)) + CR_T^D + f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) \\ &\leq f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T + CR_T^D\end{aligned}$$

where the last inequality is due to $f_T(\mathbf{x}_T) - f_T(\mathbf{x}_T) \geq 0$ and $C \geq 0$.

From the above inequality, it is easy to verify that

$$R_T^D \leq \frac{f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T}{1-C} = \frac{4\alpha D^2(f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T)}{\beta_f \epsilon^2} \leq \frac{4\alpha D^2(M + V_T)}{\beta_f \epsilon^2}$$

where the equality is due to the definition of C and the last inequality is due to the definition of M.

5. Conclusion and Future Work

In this paper, we first improve the dynamic regret bound of the OFW algorithm for smooth functions from $O(\sqrt{T(V_T + \sqrt{D_T} + 1)})$ to $O(\sqrt{T(V_T+1)})$. Second, if functions are smooth and strongly convex, we establish a tighter bound of $O(T^{1/3}(V_T+1)^{2/3})$ for OFW over strongly convex sets. Finally, for smooth and strongly convex functions with minimizers in the interior of the constraint set, we show that the dynamic regret of OFW reduces to $O(V_T+1)$, and can be further strengthened to $O(\min\{P_T, S_T, V_T\} + 1)$ by utilizing a constant number of FW iterations per round.

Note that all our results of OFW require the smoothness of functions. A natural open problem is whether our results can be extended into the non-smooth case. From our current analysis, the convergence of FW is critical for the dynamic regret of OFW. However, for non-smooth functions, FW

is not guaranteed to converge in general. Thus, it seems highly non-trivial to control the dynamic regret of OFW or its variants in the non-smooth case. Moreover, the line search rule in (7) is only valid for smooth functions. Although one may replace it with $\sigma_t = \operatorname{argmin}_{\sigma \in [0;1]} f_t((1-\sigma)\mathbf{x}_t + \sigma \mathbf{v}_t)$ in the non-smooth case, the time complexity is unclear and could be much higher than that of (7). For these reasons, we leave this problem as a future work.

Another potential limitation of this paper is that the dynamic regret defined in (2) could be too pessimistic as it compares the player against the minimizer of each round. To address this limitation, very recently, we have proposed an algorithm with an $O(T^{3/4}(P_T + 1)^{1/4})$ upper bound on the general dynamic regret defined in (3) for convex functions, by utilizing $O(\log T)$ FW iterations per round (Wang et al., 2023). However, it is still unclear whether the upper bound can be further improved without or with additional assumptions on functions and sets, which will be investigated in the future.

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Appendix A. Proof of Theorem 1

Let $R_T^D = \sum_{t=1}^T f_t(\mathbf{x}_t) - \sum_{t=1}^T f_t(\mathbf{x}_t)$. We first notice that (13) in the proof of Theorem 2 still holds for OFW. Combining it with Lemma 2, we have

$$\begin{aligned}
R_T^D &\le f_1(\mathbf{x}_1) - f_1(\mathbf{x}_1) + \sum_{t=2}^T \max_{\mathbf{x} 2 K} j f_t(\mathbf{x}) - f_{t-1}(\mathbf{x}) j + \sum_{t=2}^T (1-\sigma)(f_{t-1}(\mathbf{x}_{t-1}) - f_{t-1}(\mathbf{x}_{t-1})) \\
&\quad + \sum_{t=2}^T \left(f_{t-1}(\mathbf{x}_{t-1}) - f_t(\mathbf{x}_t) + \frac{\alpha\sigma^2 D^2}{2} \right) \\
&= f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T - (1-\sigma)(f_T(\mathbf{x}_T) - f_T(\mathbf{x}_T)) + (1-\sigma) R_T^D + \frac{\alpha(T-1)\sigma^2 D^2}{2} \\
&\le f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T + \frac{\alpha(T-1)\sigma^2 D^2}{2} + (1-\sigma) R_T^D
\end{aligned}$$

where the last inequality is due to $f_T(\mathbf{x}_T) - f_T(\mathbf{x}_T) \ge 0$ and σ 2 $[0; 1]$.

From the above inequality, it is easy to verify that

$$\sigma R_T^D \le f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T + \frac{\alpha(T-1)\sigma^2 D^2}{2}.$$

Finally, we complete this proof by dividing both sides of the above inequality with σ.

Appendix B. Proof of Theorem 3

Combining (13) with Lemma 4, we have

$$\begin{aligned}
&\sum_{t=1}^T f_t(\mathbf{x}_t) - \sum_{t=1}^T f_t(\mathbf{x}_t) \\
&\le f_1(\mathbf{x}_1) - f_1(\mathbf{x}_1) + \sum_{t=2}^T \max_{\mathbf{x} 2 K} j f_t(\mathbf{x}) - f_{t-1}(\mathbf{x}) j + \sum_{t=2}^T C_{t-1}(f_{t-1}(\mathbf{x}_{t-1}) - f_{t-1}(\mathbf{x}_{t-1})) \\
&\quad + \sum_{t=2}^T \left(f_{t-1}(\mathbf{x}_{t-1}) - f_t(\mathbf{x}_t) \right) \\
&= f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T + \sum_{t=1}^{T-1} C_t(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)) \\
&\le f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T + \sum_{t=1}^T C_t(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t))
\end{aligned}$$

where the second inequality is due to $f_T(\mathbf{x}_T) - f_T(\mathbf{x}_T) \ge 0$ and $C_T \ge 0$.

Then, for brevity, we define

$$S_T = \left\{ t \; 2 \; [T] \,\middle|\, \frac{1}{2} \le 1 - \frac{\beta_K k r f_t(\mathbf{x}_t) k_2}{8\alpha} \right\}. \tag{14}$$

From the above two equations, we have

$$\begin{aligned}&\sum_{t\in S_T}\frac{\beta_K\|\nabla f_t(\mathbf{x}_t)\|_2}{8\alpha}(f_t(\mathbf{x}_t)-f_t(\mathbf{x}_t))+\sum_{t\in[T]\setminus S_T}\frac{1}{2}(f_t(\mathbf{x}_t)-f_t(\mathbf{x}_t))\\&\le f_1(\mathbf{x}_1)-f_T(\mathbf{x}_T)+V_T\le M+V_T\end{aligned}\tag{15}$$

where the last inequality is due to the definition of M.

To analyze the first term in the left side of (15), we further note that as proved by Garber and Hazan (2015), if a function $f(\mathbf{x}):\mathcal{K}\mapsto\mathbb{R}$ is β_f-strongly convex, we have

$$\|\nabla f(\mathbf{x})\|_2\ge\sqrt{\frac{\beta_f}{2}}\sqrt{f(\mathbf{x})-f(\mathbf{x}^*)}\tag{16}$$

for any $\mathbf{x}\in\mathcal{K}$ and $\mathbf{x}^*=\operatorname{argmin}_{\mathbf{x}\in\mathcal{K}}f(\mathbf{x})$.

Because of Assumption 3, we have

$$\begin{aligned}\sum_{t\in S_T}(f_t(\mathbf{x}_t)-f_t(\mathbf{x}_t))^{3/2}&\overset{(16)}{\le}\sum_{t\in S_T}\sqrt{\frac{2}{\beta_f}}\|\nabla f_t(\mathbf{x}_t)\|_2(f_t(\mathbf{x}_t)-f_t(\mathbf{x}_t))\\&\overset{(15)}{\le}\frac{8\sqrt{2}\alpha}{\sqrt{\beta_f}\beta_K}(M+V_T).\end{aligned}\tag{17}$$

Finally, it is not hard to verify that

$$\begin{aligned}\sum_{t=1}^{T}f_t(\mathbf{x}_t)-\sum_{t=1}^{T}f_t(\mathbf{x}_t^*)&\overset{(14)}{=}\sum_{t\in S_T}(f_t(\mathbf{x}_t)-f_t(\mathbf{x}_t))+\sum_{t\in[T]\setminus S_T}(f_t(\mathbf{x}_t)-f_t(\mathbf{x}_t^*))\\&\overset{(15)}{\le}\sum_{t\in S_T}(f_t(\mathbf{x}_t)-f_t(\mathbf{x}_t))+2(M+V_T)\\&\le\left(\sum_{t\in S_T}(f_t(\mathbf{x}_t)-f_t(\mathbf{x}_t))^{3/2}\right)^{2/3}T^{1/3}+2(M+V_T)\\&\overset{(17)}{\le}\left(\frac{8\sqrt{2}\alpha}{\sqrt{\beta_f}\beta_K}(M+V_T)\right)^{2/3}T^{1/3}+2(M+V_T)\end{aligned}$$

where the second inequality is due to Hölder's inequality and $|S_T|\le T$.

Appendix C. Proof of Theorem 5

We introduce the following lemma

Lemma 6 *Under Assumptions 1, 2, 3, and 5, for any $t\in[T]$, Algorithm 2 ensures*

$$f_t(\mathbf{x}_{t+1})-f_t(\mathbf{x}_t)\le C^K(f_t(\mathbf{x}_t)-f_t(\mathbf{x}_t^*))$$

where $C=1-\frac{\beta_f r^2}{4\alpha D^2}$ and $r=\min\left\{r,\sqrt{\frac{2\alpha D^2}{\beta_f M}}\right\}$.

Combining (9) with Lemma 6, for $t = 1; \cdots ; T - 1$, we have

$$f_{t+1}(\mathbf{x}_{t+1}) - f_{t+1}(\mathbf{x}_{t+1}) \leq \max_{\mathbf{x} 2 K} j f_{t+1}(\mathbf{x}) - f_t(\mathbf{x}) j + C^K (f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)) + f_t(\mathbf{x}_t) - f_{t+1}(\mathbf{x}_{t+1}):$$

Then, similar to the proof of Theorem 4, by substituting the above inequality into (13), it is not hard to verify that

$$\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}_t) \leq f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T + C^K \left(\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}_t) \right)$$

which implies that

$$\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}_t) \leq \frac{f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T}{1 - C^K} \leq \frac{4\alpha(M + V_T)}{4\alpha - \beta_f} \tag{18}$$

where the last inequality is due to the definition of M and

$$C^K \leq C^{\frac{\ln(\beta_f = 4\alpha)}{\ln C}} = \frac{\beta_f}{4\alpha}: \tag{19}$$

To further derive dynamic regret bounds depending on P_T and S_T, we introduce nice properties of strongly convex functions and smooth functions. First, as proved by Garber and Hazan (2015), if a function $f(\mathbf{x})$ is β_f-strongly convex over K, we have

$$f(\mathbf{x}) - f(\mathbf{x}\ ) \geq \frac{\beta_f}{2} k\mathbf{x} - \mathbf{x}\ k_2^2 \tag{20}$$

for any $\mathbf{x}\ 2\ K$ and $\mathbf{x}\ = \text{argmin}_{\mathbf{x} 2 K} f(\mathbf{x})$.

Second, from Assumption 1, $f_t(\mathbf{x})$ is α-smooth over K, which satisfies that

$$\begin{aligned} f_t(\mathbf{x}) - f_t(\mathbf{x}_t) &\leq h r f_t(\mathbf{x}_t); \mathbf{x} - \mathbf{x}_t i + \frac{\alpha}{2} k\mathbf{x}_t - \mathbf{x} k_2^2 \\ &= \frac{\alpha}{2} k\mathbf{x}_t - \mathbf{x} k_2^2 \end{aligned} \tag{21}$$

for any $\mathbf{x}\ 2\ K$, where the last equality is due to the fact that Assumption 5 also implies $r f_t(\mathbf{x}_t) = \mathbf{0}$.

By using these two properties, for any $t\ 2\ [T]$, we have

$$\begin{aligned} k\mathbf{x}_{t+1} - \mathbf{x}_t k_2^2 &\overset{(20)}{\leq} \frac{2}{\beta_f} (f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t)) \leq \frac{2C^K}{\beta_f} (f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)) \\ &\overset{(21)}{\leq} \frac{\alpha C^K}{\beta_f} k\mathbf{x}_t - \mathbf{x}_t k_2^2 \overset{(19)}{\leq} \frac{1}{4} k\mathbf{x}_t - \mathbf{x}_t k_2^2 \end{aligned} \tag{22}$$

where the second inequality is due to Lemma 6.

Then, because of Assumption 2, it is not hard to verify that

$$\begin{aligned}\sum_{t=1}^{T}\|\mathbf{x}_t-\mathbf{x}_t\|_2 &\le \|\mathbf{x}_1-\mathbf{x}_1\|_2+\sum_{t=2}^{T}\|\mathbf{x}_t-\mathbf{x}_t\|_2\\ &\le D+\sum_{t=2}^{T}\|\mathbf{x}_t-\mathbf{x}_{t-1}\|_2+\sum_{t=2}^{T}\|\mathbf{x}_t-\mathbf{x}_{t-1}\|_2\\ &\overset{(22)}{\le} D+\frac{1}{2}\sum_{t=2}^{T}\|\mathbf{x}_{t-1}-\mathbf{x}_{t-1}\|_2+P_T\end{aligned} \tag{23}$$

which implies that

$$\sum_{t=1}^{T}\|\mathbf{x}_t-\mathbf{x}_t\|_2 \le 2D+2P_T. \tag{24}$$

Now, we achieve the dynamic regret bound depending on P_T as follows

$$\begin{aligned}\sum_{t=1}^{T}f_t(\mathbf{x}_t)-\sum_{t=1}^{T}f_t(\mathbf{x}_t) &\le \sum_{t=1}^{T}\langle \nabla f_t(\mathbf{x}_t),\mathbf{x}_t-\mathbf{x}_t\rangle \le \sum_{t=1}^{T}G\|\mathbf{x}_t-\mathbf{x}_t\|_2\\ &\overset{(24)}{\le} 2GD+2GP_T\end{aligned} \tag{25}$$

where the second inequality is due to the definition of G.

Furthermore, similar to (23), we also have

$$\begin{aligned}\sum_{t=1}^{T}\|\mathbf{x}_t-\mathbf{x}_t\|_2^2 &\le \|\mathbf{x}_1-\mathbf{x}_1\|_2^2+\sum_{t=2}^{T}\|\mathbf{x}_t-\mathbf{x}_t\|_2^2\\ &\le D^2+2\sum_{t=2}^{T}\|\mathbf{x}_t-\mathbf{x}_{t-1}\|_2^2+2\sum_{t=2}^{T}\|\mathbf{x}_t-\mathbf{x}_{t-1}\|_2^2\\ &\overset{(22)}{\le} D^2+\frac{1}{2}\sum_{t=2}^{T}\|\mathbf{x}_{t-1}-\mathbf{x}_{t-1}\|_2^2+2S_T\end{aligned}$$

which implies that

$$\sum_{t=1}^{T}\|\mathbf{x}_t-\mathbf{x}_t\|_2^2 \le 2D^2+4S_T. \tag{26}$$

Then, we have

$$\sum_{t=1}^{T}f_t(\mathbf{x}_t)-\sum_{t=1}^{T}f_t(\mathbf{x}_t) \overset{(21)}{\le} \frac{\alpha}{2}\sum_{t=1}^{T}\|\mathbf{x}_t-\mathbf{x}_t\|_2^2 \overset{(26)}{\le} \alpha D^2+2\alpha S_T. \tag{27}$$

Finally, by combining (18), (25), and (27), we complete this proof.

Appendix D. Proof of Lemma 2

We first note that (9) and (11) in the proof of Lemma 3 still hold for OFW. So, we only need to analyze $f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t)$. According to (4), we have

$$\mathbf{x}_{t+1} - \mathbf{x}_t = \sigma(\mathbf{v}_t - \mathbf{x}_t).$$

Then, combining this equality with the smoothness of $f_t(\mathbf{x})$, we have

$$\begin{aligned} f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t) &\le f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t) + \sigma\langle \nabla f_t(\mathbf{x}_t), \mathbf{v}_t - \mathbf{x}_t\rangle + \frac{\alpha\sigma^2}{2}\|\mathbf{v}_t - \mathbf{x}_t\|_2^2 \\ &\le f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t) + \sigma\langle \nabla f_t(\mathbf{x}_t), \mathbf{v}_t - \mathbf{x}_t\rangle + \frac{\alpha\sigma^2 D^2}{2} \\ &\overset{(11)}{\le} (1-\sigma)(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)) + \frac{\alpha\sigma^2 D^2}{2} \end{aligned} \tag{28}$$

where the second inequality is due to Assumption 2.

Finally, we complete this proof by substituting (28) into (9).

Appendix E. Proof of Lemma 4

Following the proof of Lemma 3, we still utilize (9) and (10). The main difference is to establish a tighter bound for $f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t)$ by further using Assumption 4. Specifically, we first define

$$\mathbf{c}_t = \frac{1}{2}(\mathbf{x}_t + \mathbf{v}_t), \mathbf{w}_t \in \operatorname*{argmin}_{\|\mathbf{w}\|_2 \le 1} \langle \nabla f_t(\mathbf{x}_t), \mathbf{w}\rangle, \text{ and } \mathbf{v}_t^0 = \mathbf{c}_t + \frac{\beta_K}{8}\|\mathbf{x}_t - \mathbf{v}_t\|_2^2 \mathbf{w}_t.$$

According to Assumption 4, it is easy to verify that

$$\mathbf{v}_t^0 \in K.$$

Moreover, because of $\mathbf{v}_t \in \operatorname{argmin}_{\mathbf{x}\in K}\langle \nabla f_t(\mathbf{x}_t), \mathbf{x}\rangle$ in Algorithm 1, we have

$$\begin{aligned} \langle \nabla f_t(\mathbf{x}_t), \mathbf{v}_t - \mathbf{x}_t\rangle &\le \langle \nabla f_t(\mathbf{x}_t), \mathbf{v}_t^0 - \mathbf{x}_t\rangle \\ &= \frac{1}{2}\langle \nabla f_t(\mathbf{x}_t), \mathbf{v}_t - \mathbf{x}_t\rangle + \frac{\beta_K\|\mathbf{x}_t - \mathbf{v}_t\|_2^2}{8}\langle \nabla f_t(\mathbf{x}_t), \mathbf{w}_t\rangle \\ &\overset{(11)}{\le} -\frac{1}{2}(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)) + \frac{\beta_K\|\mathbf{x}_t - \mathbf{v}_t\|_2^2}{8}\langle \nabla f_t(\mathbf{x}_t), \mathbf{w}_t\rangle \\ &= -\frac{1}{2}(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)) - \frac{\beta_K\|\mathbf{x}_t - \mathbf{v}_t\|_2^2}{8}\|\nabla f_t(\mathbf{x}_t)\|_2 \end{aligned} \tag{29}$$

where the last equality is due to

$$\langle \nabla f_t(\mathbf{x}_t), \mathbf{w}_t\rangle = \min_{\|\mathbf{w}\|_2 \le 1}\langle \nabla f_t(\mathbf{x}_t), \mathbf{w}\rangle = -\|\nabla f_t(\mathbf{x}_t)\|_2. \tag{30}$$

Substituting (29) into (10), for all possible $\sigma \in [0, 1]$, we have

$$\begin{aligned} & f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t) \\ &\le \left(1 - \frac{\sigma}{2}\right)(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)) + \frac{\alpha\sigma^2}{2}\|\mathbf{v}_t - \mathbf{x}_t\|_2^2 - \frac{\sigma\,\beta_K\|\mathbf{x}_t - \mathbf{v}_t\|_2^2}{8}\|\nabla f_t(\mathbf{x}_t)\|_2 \\ &= \left(1 - \frac{\sigma}{2}\right)(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)) + \left(\alpha\sigma^2 - \frac{\sigma\,\beta_K\|\nabla f_t(\mathbf{x}_t)\|_2}{4}\right)\frac{\|\mathbf{x}_t - \mathbf{v}_t\|_2^2}{2}. \end{aligned} \tag{31}$$

Then, if $\alpha \leq \frac{\beta_K k r f_t(\mathbf{x}_t)k_2}{4}$, by combining $\sigma = 1$ with (31), we have

$$f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t) \leq \frac{1}{2}(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)): \tag{32}$$

Otherwise, by combining $\sigma = \frac{\beta_K k r f_t(\mathbf{x}_t)k_2}{4\alpha}$ with (31), we have

$$f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t) \leq \left(1 - \frac{\beta_K k r f_t(\mathbf{x}_t)k_2}{8\alpha}\right)(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)): \tag{33}$$

Combining (32) and (33) with the definition of C_t, we have

$$f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t) \leq C_t(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)): \tag{34}$$

Finally, we complete this proof by substituting (34) into (9).

Appendix F. Proof of Lemma 5

Note that Algorithm 2 with $K = 1$ reduces to Algorithm 1. Therefore, Lemma 6 with $K = 1$ also holds for Algorithm 1, which implies that

$$f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t) \leq C(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)):$$

By substituting the above inequality into (9), we complete this proof.

Appendix G. Proof of Lemma 6

This lemma can be derived from the convergence rate of FW under Assumptions 1, 2, 3, and 5 (see Section 4.2 of Garber and Hazan (2015) for the details). For the sake of completeness, we include the detailed proof here.

We first define

$$\mathbf{w}_t^i \in \operatorname*{argmin}_{\|\mathbf{w}\|_2 \leq 1} \langle r f_t(\mathbf{z}_{t+1}^i), \mathbf{w} \rangle \text{ and } \mathbf{v}_t^i = \mathbf{x}_t + \tau \mathbf{w}_t^i$$

where $\mathbf{v}_t^i$ also belongs to K since $\mathbf{x}_t$ lies in the interior of K and $\tau \leq r$.

Then, because of $\mathbf{v}_t^i \in \operatorname{argmin}_{\mathbf{x} \in K} \langle r f_t(\mathbf{z}_{t+1}^i), \mathbf{x} \rangle$ in Algorithm 2, we have

$$\begin{aligned} \langle r f_t(\mathbf{z}_{t+1}^i), \mathbf{v}_t^i - \mathbf{z}_{t+1}^i \rangle &\leq \langle r f_t(\mathbf{z}_{t+1}^i), \mathbf{v}_t^i - \mathbf{z}_{t+1}^i \rangle \\ &= \langle r f_t(\mathbf{z}_{t+1}^i), \mathbf{x}_t - \mathbf{z}_{t+1}^i \rangle + \tau \langle r f_t(\mathbf{z}_{t+1}^i), \mathbf{w}_t^i \rangle \\ &\leq -\tau k r f_t(\mathbf{z}_{t+1}^i)k_2 \end{aligned} \tag{35}$$

where the last inequality is due to

$$\langle r f_t(\mathbf{z}_{t+1}^i), \mathbf{w}_t^i \rangle = \min_{\|\mathbf{w}\|_2 \leq 1} \langle r f_t(\mathbf{z}_{t+1}^i), \mathbf{w} \rangle = -k r f_t(\mathbf{z}_{t+1}^i)k_2$$

and

$$\langle r f_t(\mathbf{z}_{t+1}^i), \mathbf{x}_t - \mathbf{z}_{t+1}^i \rangle \leq f_t(\mathbf{x}_t) - f_t(\mathbf{z}_{t+1}^i) \leq 0:$$

Combining the smoothness of $f_t(\mathbf{x})$ with $\mathbf{z}_{t+1}^{i+1} - \mathbf{z}_{t+1}^{i} = \sigma_t^i(\mathbf{v}_t^i - \mathbf{z}_{t+1}^i)$ in Algorithm 2, for all possible $\sigma \in [0,1]$, we have

$$\begin{aligned} f_t(\mathbf{z}_{t+1}^{i+1}) - f_t(\mathbf{x}_t) &\le f_t(\mathbf{z}_{t+1}^{i}) - f_t(\mathbf{x}_t) + \sigma_t^i \langle \nabla f_t(\mathbf{z}_{t+1}^{i}), \mathbf{v}_t^i - \mathbf{z}_{t+1}^i \rangle + \frac{\alpha(\sigma_t^i)^2}{2}\|\mathbf{v}_t^i - \mathbf{z}_{t+1}^i\|_2^2 \\ &\le f_t(\mathbf{z}_{t+1}^{i}) - f_t(\mathbf{x}_t) + \sigma \langle \nabla f_t(\mathbf{z}_{t+1}^{i}), \mathbf{v}_t^i - \mathbf{z}_{t+1}^i \rangle + \frac{\alpha\sigma^2}{2}\|\mathbf{v}_t^i - \mathbf{z}_{t+1}^i\|_2^2 \\ &\le f_t(\mathbf{z}_{t+1}^{i}) - f_t(\mathbf{x}_t) + \sigma \langle \nabla f_t(\mathbf{z}_{t+1}^{i}), \mathbf{v}_t^i - \mathbf{z}_{t+1}^i \rangle + \frac{\alpha\sigma^2 D^2}{2} \\ &\overset{(35)}{\le} f_t(\mathbf{z}_{t+1}^{i}) - f_t(\mathbf{x}_t) - \sigma \kappa \|\nabla f_t(\mathbf{z}_{t+1}^{i})\|_2 + \frac{\alpha\sigma^2 D^2}{2} \\ &\overset{(16)}{\le} f_t(\mathbf{z}_{t+1}^{i}) - f_t(\mathbf{x}_t) - \sigma \kappa \sqrt{\frac{\beta_f (f_t(\mathbf{z}_{t+1}^{i}) - f_t(\mathbf{x}_t))}{2}} + \frac{\alpha\sigma^2 D^2}{2} \end{aligned} \tag{36}$$

where the second inequality is due to the line search rule in Algorithm 2 and the third inequality is due to Assumption 2.

Moreover, we set σ as

$$0 \le \sigma = \frac{\kappa\sqrt{\beta_f (f_t(\mathbf{z}_{t+1}^{i}) - f_t(\mathbf{x}_t))}}{\beta_f \alpha D^2} \le \frac{\kappa\sqrt{\beta_f M}}{\beta_f \alpha D^2} \le 1$$

where the last inequality is due to the definition of κ.

Combining this σ with (36), we have

$$f_t(\mathbf{z}_{t+1}^{i+1}) - f_t(\mathbf{x}_t) \le f_t(\mathbf{z}_{t+1}^{i}) - f_t(\mathbf{x}_t) - \frac{\kappa^2 \beta_f (f_t(\mathbf{z}_{t+1}^{i}) - f_t(\mathbf{x}_t))}{4\alpha D^2} = C(f_t(\mathbf{z}_{t+1}^{i}) - f_t(\mathbf{x}_t))$$

where the last equality is due to the definition of C.

Finally, from the above inequality, it is easy to verify that

$$\begin{aligned} f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t) &= f_t(\mathbf{z}_{t+1}^{K}) - f_t(\mathbf{x}_t) \le C^K (f_t(\mathbf{z}_{t+1}^{0}) - f_t(\mathbf{x}_t)) \\ &= C^K (f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t)). \end{aligned}$$

Appendix H. Dynamic Regret of OGD for Smooth Functions

As discussed in Section 2, our $O(\sqrt{T(V_T + 1)})$ dynamic regret for smooth functions is not achieved by previous studies even using projection-based algorithms. Here, we notice that OGD (Zinkevich, 2003) actually also enjoys this bound. Specifically, the detailed procedures of OGD are summarized in Algorithm 3, where the step size is simply set as $1/\alpha$ by following the common choice in offline optimization (Bubeck, 2015). Then, we establish an $O(\sqrt{T(V_T + 1)})$ dynamic regret bound for Algorithm 3 in the following theorem.

Theorem 6 *Under Assumptions 1 and 2, Algorithm 3 ensures*

$$\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}_t) \le M + V_T + \sqrt{2\alpha D^2 (T-1)(M + V_T)}$$

where $V_T = \sum_{t=2}^{T} \max_{\mathbf{x} \in \mathcal{K}} \|f_t(\mathbf{x}) - f_{t-1}(\mathbf{x})\|$ *and* $M = \max_{t \in [T], \mathbf{x} \in \mathcal{K}} \{2\|f_t(\mathbf{x})\|\}$.

Algorithm 3 Online Gradient Descent

1: **Initialization:** $\mathbf{x}_1 \in K$
2: **for** $t = 1, \cdots, T$ **do**
3: Compute $\mathbf{x}'_{t+1} = \mathbf{x}_t - \frac{1}{\alpha} \nabla f_t(\mathbf{x}_t)$
4: Compute $\mathbf{x}_{t+1} = \operatorname{argmin}_{\mathbf{x} \in K} \|\mathbf{x} - \mathbf{x}'_{t+1}\|_2^2$
5: **end for**

Proof We start this proof by introducing the following lemma.

Lemma 7 *(Lemma 3.6 of Bubeck (2015)) Let $f(\cdot) : K \mapsto \mathbb{R}$ be a convex and α-smooth function, and $\mathbf{x}' = \operatorname{argmin}_{\mathbf{x} \in K} \left\| \mathbf{x} - \frac{1}{\alpha} \nabla f(\mathbf{x}) \right\|_2^2$. Then, for any $\mathbf{x} \in K, \mathbf{y} \in K$, we have*

$$f(\mathbf{x}') - f(\mathbf{y}) \leq \alpha \langle \mathbf{x} - \mathbf{x}', \mathbf{x} - \mathbf{y} \rangle - \frac{\alpha}{2} \|\mathbf{x} - \mathbf{x}'\|_2^2.$$

From Lemma 7, it is easy to verify that

$$f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t) \leq -\frac{\alpha}{2} \|\mathbf{x}_{t+1} - \mathbf{x}_t\|_2^2 \tag{37}$$

and

$$f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t) \leq \alpha \|\mathbf{x}_t - \mathbf{x}_{t+1}\|_2 \|\mathbf{x}_t - \mathbf{x}_t\|_2. \tag{38}$$

Then, combining with Assumption 2, we have

$$\begin{aligned} f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t) &\overset{(37)}{\leq} f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t) - \frac{\alpha}{2} \|\mathbf{x}_{t+1} - \mathbf{x}_t\|_2^2 \\ &\overset{(38)}{\leq} f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t) - \frac{(f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t))^2}{2\alpha \|\mathbf{x}_t - \mathbf{x}_t\|_2^2} \\ &\leq f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t) - \frac{(f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t))^2}{2\alpha D^2}. \end{aligned} \tag{39}$$

Moreover, we notice that

$$\begin{aligned} &\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}_t) \\ &= f_1(\mathbf{x}_1) - f_1(\mathbf{x}_1) + \sum_{t=1}^{T-1} \left(f_{t+1}(\mathbf{x}_{t+1}) - f_{t+1}(\mathbf{x}_{t+1}) \right) \\ &\overset{(9)}{\leq} f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T + \sum_{t=1}^{T-1} (f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t)) \\ &\overset{(39)}{\leq} f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T + \sum_{t=1}^{T-1} \left(f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t) - \frac{(f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t))^2}{2\alpha D^2} \right). \end{aligned} \tag{40}$$

From the above inequality, we first have

$$\sum_{t=1}^{T-1} (f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t))^2 \leq 2\alpha D^2 (f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T) \leq 2\alpha D^2 (M + V_T). \tag{41}$$

Then, we further have

$$\begin{aligned}\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}_t) &\overset{(40)}{\leq} f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T + \sum_{t=1}^{T-1} (f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t)) \\ &\leq f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T + \sqrt{(T-1)\sum_{t=1}^{T-1} (f_t(\mathbf{x}_{t+1}) - f_t(\mathbf{x}_t))^2} \quad (42) \\ &\overset{(41)}{\leq} f_1(\mathbf{x}_1) - f_T(\mathbf{x}_T) + V_T + \sqrt{2\alpha D^2 (T-1)(M + V_T)} \\ &\leq M + V_T + \sqrt{2\alpha D^2 (T-1)(M + V_T)}\end{aligned}$$

where the second inequality is due to Cauchy–Schwarz inequality.

■