Supplementary Material: Online Bandit Learning for a Special Class of Non-convex Losses

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Proof of Proposition 1

The proof is similar to that of Lemma 4 in (Zhang, Yi, and Jin 2014). First, we have

$$\begin{aligned} & \mathbf{u}^\top \mathrm{E}_{t-1}\left[-\frac{Z_t c_t}{\|\mathbf{v}_t\|_2}\mathbf{v}_t\right] \\ \overset{(1)}{=} & - \mathrm{E}_{t-1}\left[f\left(\mathbf{u}^\top \frac{\mathbf{v}_t}{\|\mathbf{v}_t\|_2}\right)\mathbf{u}^\top \frac{\mathbf{v}_t}{\|\mathbf{v}_t\|_2}\right] \\ \overset{(3),(5)}{=} & \quad : \end{aligned}$$

Then, consider any vector $\mathbf{u} \perp \mathbf{u}$. We have

$$\begin{aligned} & \mathbf{u}^\top \mathrm{E}_{t-1}\left[-\frac{Z_t c_t}{\|\mathbf{v}_t\|_2}\mathbf{v}_t\right] \\ \overset{(1)}{=} & - \mathrm{E}_{t-1}\left[f\left(\mathbf{u}^\top \frac{\mathbf{v}_t}{\|\mathbf{v}_t\|_2}\right)\mathbf{u}^\top \frac{\mathbf{v}_t}{\|\mathbf{v}_t\|_2}\right] \\ = & - \mathrm{E}_{t-1}\left[f\left(\frac{1}{\sqrt{\ _1^2 + \cdots + \ _d^2}}\right)\frac{\|\mathbf{u}\|_2 \ _2}{\sqrt{\ _1^2 + \cdots + \ _d^2}}\right] \end{aligned}$$

where $_1; : : : ; \ _n$ are n independent standard Gaussian variables. Notice that $f\left(\frac{\alpha_1}{\sqrt{\alpha_1^2+\cdots+\alpha_d^2}}\right)\frac{\|\mathbf{u}\|_2\alpha_2}{\sqrt{\alpha_1^2+\cdots+\alpha_d^2}}$ is an odd function of $_2$, so its expectation with respect to $_2$ must be 0. Thus, we have

$$\mathbf{u}^\top \mathrm{E}_{t-1}\left[-\frac{Z_t c_t}{\|\mathbf{v}_t\|_2}\mathbf{v}_t\right] = 0:$$

Then, it is easy to prove Proposition 1 by contradiction.

Proof of Theorem 3

The following lemma extends Lemma 2 to the general case when each $\mathbf{u}_t$ is a different vector.

Lemma 4. *We have*

$$f(\mathbf{x}_t^\top \mathbf{u}_t) - \min_{\|\mathbf{x}\|\leq 1} f(\mathbf{x}^\top \mathbf{u}_t) \leq$$

$$\begin{cases} 2B; & \mathrm{E}_t \quad {}_T \text{if } t = 1; \quad T \quad T\ i/ \\ \frac{2L}{\gamma\eta\rho_T(t-1)}\left\|\sum_{i=1}^{t-1} \ _i\right\|_2 + 2L\|\mathbf{u}_t - & \end{cases}$$

Following a simple geometric argument, we have

$$\left\| \frac{\mathbf{w}_t}{\|\mathbf{w}_t\|} - \frac{\mathbf{u}_{t-1}}{\|\mathbf{u}_{t-1}\|_2} \right\|_2 \leq \frac{2}{(t-1)\|\mathbf{u}_{t-1}\|_2} \left\| \sum_{i=1}^{t-1} \ _i \right\|_2 ;$$

implying

$$\begin{aligned} & f(\mathbf{x}_t^\top \mathbf{u}) - \min_{\|\mathbf{x}\| \leq 1} f(\mathbf{x}^\top \mathbf{u}) \\ \overset{(18)}{\leq} & \frac{2L}{(t-1)\|\mathbf{u}_{t-1}\|_2} \left\| \sum_{i=1}^{t-1} \ _i \right\|_2 + L \left\| \mathbf{u}_t - \frac{\mathbf{u}_{t-1}}{\|\mathbf{u}_{t-1}\|_2} \right\|_2 \\ \leq & \frac{2L}{\ _T(t-1)} \left\| \sum_{i=1}^{t-1} \ _i \right\|_2 + 2L\|\mathbf{u}_t - \mathbf{u}_{t-1}\|_2 : \end{aligned}$$

Proof of (7)

We provide the proof because our definition of $\mathbf{u}_t$ is slightly different from the one in (Hazan and Kale 2010).

First, we have

$$\begin{aligned} & \sum_{t=2}^{T} \|\mathbf{u}_t - \mathbf{u}_{t-1}\|_2^2 - \sum_{t=2}^{T} \|\mathbf{u}_t - \mathbf{u}_t\|_2^2 \\ \leq & \sum_{t=2}^{T} 2\langle \mathbf{u}_t - \mathbf{u}_{t-1}; \mathbf{u}_t - \mathbf{u}_{t-1} \rangle \\ = & \sum_{t=2}^{T} 2 \left\langle \mathbf{u}_t - \mathbf{u}_{t-1}; \frac{(t-1)\mathbf{u}_{t-1} + \mathbf{u}_t}{t} - \mathbf{u}_{t-1} \right\rangle \\ = & 2\sum_{t=2}^{T} \frac{\|\mathbf{u}_t - \mathbf{u}_{t-1}\|_2^2}{t} \leq 4 \sum_{t=2}^{T} \frac{\|\mathbf{u}_t - \mathbf{u}_{t-1}\|_2}{t} \end{aligned} \tag{19}$$

where the first inequality is due to the property of convexity and the second inequality comes from our assumption that $\|\mathbf{u}_t\|_2 = 1, \forall t \in [T]$. We then bound $\sum_{t=2}^{T} \frac{\|\mathbf{u}_t - \mathbf{u}_{t \ 1}\|_2}{t}$ as follows.

$$\begin{aligned} & \sum_{t=2}^{T} \frac{\|\mathbf{u}_t - \mathbf{u}_{t-1}\|_2}{t} = \sum_{t=2}^{T} \frac{1}{t} \left\| \mathbf{u}_t - \frac{1}{t-1} \sum_{i=1}^{t-1} \mathbf{u}_i \right\|_2 \\ = & \sum_{t=2}^{T} \frac{1}{t} \left\| (\mathbf{u}_t - \mathbf{u}_T) - \frac{1}{t-1} \sum_{i=1}^{t-1} (\mathbf{u}_i - \mathbf{u}_T) \right\|_2 \\ \leq & \sum_{t=2}^{T} \frac{1}{t} \|\mathbf{u}_t - \mathbf{u}_T\|_2 + \sum_{t=1}^{T-1} \|\mathbf{u}_t - \mathbf{u}_T\|_2 \sum_{i=t}^{T-1} \frac{1}{i(i+1)} \\ \leq & \sum_{t=2}^{T} \frac{1}{t} \|\mathbf{u}_t - \mathbf{u}_T\|_2 + \sum_{t=1}^{T-1} \frac{1}{t} \|\mathbf{u}_t - \mathbf{u}_T\|_2 \\ \leq & 2\sum_{t=1}^{T} \frac{1}{t} \|\mathbf{u}_t - \mathbf{u}_T\|_2 \\ \leq & 2\sqrt{\sum_{t=1}^{T} \|\mathbf{u}_t - \mathbf{u}_T\|_2^2} \sqrt{\sum_{t=1}^{T} \frac{1}{t^2}} \leq 3\sqrt{\sum_{t=1}^{T} \|\mathbf{u}_t - \mathbf{u}_T\|_2^2} : \end{aligned} \tag{20}$$

$\mathbf{u}_t k_2$ $\leq 1 \|_2^2 -$

Combining (19) with (20), we have

$$\sum_{t=2}^{T} \|\mathbf{u}_t - \mathbf{u}_{t-1}\|_2^2 \in$$

$_2 \leq 3\sqrt{}$